A NOTE ON k-VERY AMPLENESS OF LINE BUNDLES ON GENERAL BLOW-UPS OF HYPERELLIPTIC SURFACES

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Abstract. We study k-very ampleness of line bundles on blow-ups of hyperelliptic surfaces at r very general points. We obtain a numerical condition on the number of points for which a line bundle on the blow-up of a hyperelliptic surface at these r points gives an embedding of order k.

1. Introduction

M.C. Beltrametti, P. Francia and A.J. Sommese introduced and studied the concepts of higher order embeddings: k-spandness, k-very ampleness and k-jet ampleness of polarised varieties in a series of papers, see [BeFS1989], [BeS1988], [BeS1993]. The problem of k-very ampleness on certain surfaces was studied by many authors. M. Mella and M. Palleschi in [MP1993] proved the necessary and sufficient condition for a line bundle on any hyperelliptic surface to be k-very ample. Such a condition for any Del Pezzo surface was given by S. Di Rocco in [DR1996]. Th. Bauer and T. Szemberg in [BaSz1997] provided a criterion for k-very ampleness of a line bundle on an abelian surface.

In [SzT-G2002] T. Szemberg and H. Tutaj-Gasińska established a condition on the number of points for which a line bundle is k-very ample on a general blow-up of the projective plane. H. Tutaj-Gasińska in [T-G2002] gave a condition for k-very ampleness of a line bundle on a general blow-up of an abelian surface, and in [T-G2005] — on general blow-ups of elliptic quasi-bundles.

Recently, W. Alagal and A. Maciocia in [AMa2014] study critical k-very ampleness on abelian surfaces, i.e. consider the critical value of k for which a line bundle is k-very ample but not $(k+1)$-very ample.

We come back to the classical question on the number of points for which a line bundle on a general blow-up of a surface is k-very ample. We consider blow-ups of hyperelliptic surfaces as such case has not been an object of study before.

2. Notation and auxiliary results

Let us set up the notation and basic definitions. We work over the field of complex numbers $\mathbb{C}$. We consider only smooth reduced and irreducible projective varieties. By $D_1 \equiv D_2$ we denote the numerical equivalence of divisors D_1 and D_2. By a curve we understand an irreducible subvariety of dimension 1. In the notation we follow [Laz2004].

We recall the definition of k-very ampleness.

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Let X be a smooth projective variety of dimension n. Let L be a line bundle on X, and let $x \in X$.

Definition 2.1. We say that a line bundle L is k-very ample if for every 0-dimensional subscheme $Z \subset X$ of length $k+1$ the restriction map

$$H^0(X, L) \longrightarrow H^0(X, L \otimes \mathcal{O}_Z)$$

is surjective.

In the other words k-very ampleness means that the subschemes of length at most $k+1$ impose independent conditions on global sections of L.

We also recall the definition of the multi-point Seshadri constant.

Let $x_1, \ldots, x_r \in X$ be pairwise distinct points.

Definition 2.2. The multi-point Seshadri constant of L at $x_1, \ldots, x_r$ is the real number

$$\varepsilon(L, x_1, \ldots, x_r) = \inf \left\{ \frac{LC}{\sum_{i=1}^r \operatorname{mult}_{x_i} C} : \ \{x_1, \ldots, x_r\} \cap C \neq \emptyset \right\},$$

where the infimum is taken over all irreducible curves $C \subset X$ passing through at least one of the points $x_1, \ldots, x_r$.

If $\pi \colon \widetilde{X} \longrightarrow X$ is the blow-up of X at $x_1, \ldots, x_r$, and $E_1, \ldots, E_r$ are exceptional divisors of the blow-up, then equivalently the Seshadri constant may be defined as (see e.g. [Laz2004] vol. I, Proposition 5.1.5):

$$\varepsilon(L, x_1, \ldots, x_r) = \sup \left\{ \varepsilon : \ \pi^* L - \varepsilon \sum_{i=1}^r E_i \text{ is nef} \right\}.$$

Now let us recall the definition of a hyperelliptic surface.

Definition 2.3. A hyperelliptic surface S (sometimes called bielliptic) is a surface with Kodaira dimension equal to 0 and irregularity $q(S) = 1$.

Alternatively ([Bea1996], Definition VI.19), a surface S is hyperelliptic if $S \cong (A \times B)/G$, where A and B are elliptic curves, and G is an abelian group acting on A by translation and acting on B, such that A/G is an elliptic curve and $B/G \cong \mathbb{P}^1$; G acts on $A \times B$ coordinatewise. Hence we have the following situation:

$$\begin{array}{ccc} S \cong (A \times B)/G & \xrightarrow{\Phi} & A/G \\ {\scriptstyle\Psi}\big\downarrow & & \\ B/G \cong \mathbb{P}^1 & & \end{array}$$

where Φ and Ψ are natural projections.

Hyperelliptic surfaces were classified at the beginning of 20th century by G. Bagnera and M. de Franchis in [BF1907], and independently by F. Enriques i F. Severi in [ES1909-10]. They showed that there are seven non-isomorphic types of hyperelliptic surfaces. Those types are characterised by the action of G on $B \cong \mathbb{C}/(\mathbb{Z}\omega \oplus \mathbb{Z})$ (for details see e.g. [Bea1996], VI.20). The canonical divisor K_S of any hyperelliptic surface is numerically trivial.

In 1990 F. Serrano in [Se1990], Theorem 1.4, characterised the group of classes of numerically equivalent divisors $\mathrm{Num}(S)$ for each of the surface's type:

Theorem 2.4 (Serrano)**.** *A basis of the group* $\mathrm{Num}(S)$ *for each of the hyperelliptic surface's type and the multiplicities of the singular fibres in each case are the following:*

Type of a hyperelliptic surface	G	$m_1, \ldots, m_s$	*Basis of* $\mathrm{Num}(S)$
1	$\mathbb{Z}_2$	$2,2,2,2$	$A/2, B$
2	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$2,2,2,2$	$A/2, B/2$
3	$\mathbb{Z}_4$	$2,4,4$	$A/4, B$
4	$\mathbb{Z}_4 \times \mathbb{Z}_2$	$2,4,4$	$A/4, B/2$
5	$\mathbb{Z}_3$	$3,3,3$	$A/3, B$
6	$\mathbb{Z}_3 \times \mathbb{Z}_3$	$3,3,3$	$A/3, B/3$
7	$\mathbb{Z}_6$	$2,3,6$	$A/6, B$

Let $\mu = \mathrm{lcm}\{m_1, \ldots, m_s\}$ and let $\gamma = |G|$. Given a hyperelliptic surface, its basis of $\mathrm{Num}(S)$ consists of divisors A/μ and $(\mu/\gamma)\,B$. We say that L is a line bundle of type (a,b) on a hyperelliptic surface if $L \equiv a \cdot A/\mu + b \cdot (\mu/\gamma)B$. In $\mathrm{Num}(S)$ we have that $A^2 = 0$, $B^2 = 0$, $AB = \gamma$.

The following proposition holds:

Proposition 2.5 (see [Se1990], Lemma 1.3)**.** *Let D be a divisor of type (a,b) on a hyperelliptic surface S. Then*

$$D \text{ is ample if and only if } a > 0 \text{ and } b > 0.$$

Now we recall the criterion for a line bundle on a surface to be k-very ample, obtained by M. Beltrametti and A. Sommese in [BeS1988].

Theorem 2.6 (Beltrametti, Sommese)**.** *Let S be a smooth projective surface. Let L be a nef line bundle on S such that $L^2 \geq 4k+5$.*

Then either $K_S + L$ is k-very ample or there exists an effective divisor D satisfying the following conditions:

(1) *$L - 2D$ is $\mathbb{Q}$-effective, i.e. there exists an integer $m > 0$ such that $|m(L-2D)| \neq \emptyset$.*

(2) *D contains a subscheme Z of length $k+1$ such that the map*

$$H^0(K_S \otimes L) \longrightarrow H^0(K_S \otimes L \otimes \mathcal{O}_Z)$$

is not surjective.

(3) $LD - k - 1 \leq D^2 < \frac{LD}{2} < k+1.$

M. Mella and M. Palleschi in [MP1993] fully characterised k-very ampleness of line bundles on hyperelliptic surfaces. For an ample line bundle $L \equiv (a,b)$ they give necessary and sufficient numerical conditions on a and b for each hyperelliptic surface's type.

We will use the sufficient condition for k-very ampleness of a line bundle on a hyperelliptic surface that is implied by [MP1993], Theorems 3.2-3.4:

Proposition 2.7 (Mella, Palleschi)**.** *Let S be a hyperelliptic surface. Let $L \equiv (a,b)$ be an ample line bundle on S. Let $k \in \mathbb{N}$.*

If $a \geq k+2$ and $b \geq k+2$ then L is k-very ample.

In the next section we will prove a condition on the number r for which a pull-back of a d-very ample line bundle on a hyperelliptic surface is k-very ample on the blow-up of this surface at r very general points.

3. Main result

We study k-very ampleness for $k \geq 2$. Case $k = 1$ for a blow-up of a smooth projective surface was considered by M. Coppens, see [Co1995], Theorem 2. Namely, Coppens proved that on a blow-up of a smooth projective surface at r points in very general position a line bundle $M = \pi^*(mL) - \sum_{i=1}^{r} E_i$, where L is an ample line bundle, is 1-very ample (i.e. very ample) if $m \geq 7$ and $r \leq h^0(mL) - 7$. Even if we proved Theorem 3.1 for $k = 1$, we would get a weaker result than Coppens.

Our main result is the following

Theorem 3.1. *Let S be a hyperelliptic surface. Let $k \geq 2$, and let $d > (k+1)^2$. Let $L_S \equiv (a, b)$ a line bundle on S with $a \geq d+2$ and $b \geq d+2$.*

Let $r \geq 2$. Let $\pi \colon \widetilde{S} \longrightarrow S$ be the blow-up of S at r points in very general position where

$$r \leq 0.887 \cdot \frac{L_S^2}{(k+1)^2}.$$

Then a line bundle $L = \pi^* L_S - k\sum_{i=1}^{r} E_i$ is k-very ample on $\widetilde{S}$.

Our proof is based on H. Tutaj-Gasińska's ideas from [T-G2005], Theorem 11. We get a more accurate estimation on the admissible number of points r than in [T-G2005]. This is caused by the fact that for hyperelliptic surfaces we have better estimation of the multi-point Seshadri constants than for arbitrary elliptic quasi-bundle, and on specifics of hyperelliptic surfaces among elliptic fibrations.

Moreover, assuming that $r \leq c \cdot \frac{L_S^2}{(k+1)^2}$ we carefully analysed the conditions for a constant c to be a maximal possible constant satisfying all conditions imposed by the proof, with any $\delta > 0$. The key restriction for the upper bound of c is given by inequalities (3.1) and (3.2). The constant 0.887 is computed to be a round down to the third decimal place of the maximal c satisfying all conditions appearing in the proof.

Proof. On hyperelliptic surfaces $K_S \equiv 0$, hence $L_S \equiv L_S - K_S \equiv (a, b)$. Obviously, $L_S^2 = 2ab \geq 2(d+2)^2 \geq \left((k+1)^2 + 3\right)^2$. We prove k-very ampleness of $L = \pi^* L_S - k\sum_{i=1}^{r} E_i$, applying Theorem 2.6 to the line bundle

$$N = L - K_{\widetilde{S}} \equiv \pi^* L_S - (k+1)\sum_{i=1}^{r} E_i.$$

In the two consecutive lemmas we check that the assumptions of Theorem 2.6 are satisfied, i.e. that $N^2 \geq 4k+5$ and that N is a nef line bundle (we prove that N is in fact ample). Finally, we show that there does not exist an effective divisor D satisfying condition (3) of Theorem 2.6.

Lemma 3.2. *With the notation above*

$$N^2 \geq 4k+5.$$

Proof of the lemma. We estimate: $N^2 = \left(\pi^* L_S - (k+1)\sum_{i=1}^r E_i\right)^2 = L_S^2 - (k+1)^2 r \geq L_S^2 - 0.887 \cdot \frac{L_S^2}{(k+1)^2} \cdot (k+1)^2 = 0.113 \cdot L_S^2 \geq 0.113 \cdot 2\left((k+1)^2+3\right)^2 = 0.113 \cdot 2(k^4+4k^3+12k^2+16k+16) \geq 0.113 \cdot 2 \cdot 16(4k+1) \geq 14k+3 \geq 4k+5.$ □

Lemma 3.3. *N is ample.*

Proof of the lemma. By [Fa2015], Theorem 3.6, we have $\varepsilon(L_S, r) \geq \sqrt{\frac{L_S^2}{r}}\sqrt{1-\frac{1}{8r}}$. We will prove that

$$(\star) \qquad \sqrt{\frac{L_S^2}{r}}\sqrt{1-\frac{1}{8r}} > k+1+\delta$$

where $\delta > 0$. Applying an equivalent definition of r-point Seshadri constant we will get an assertion of the lemma.

It is enough to show that $(\star)$ holds for the maximal admissible r, i.e. for $r = 0.887 \cdot \frac{L_S^2}{(k+1)^2}$. We ask whether

$$\sqrt{\frac{8 \cdot 0.887 \cdot \frac{L_S^2}{(k+1)^2} L_S^2 - L_S^2}{8 \cdot \left(0.887 \cdot \frac{L_S^2}{(k+1)^2}\right)^2}} > k+1+\delta$$

$$\frac{k+1}{0.887}\sqrt{0.887 - \frac{(k+1)^2}{8 \cdot L_S^2}} > k+1+\delta$$

It suffices to check that

$$(k+1)\left(\frac{1}{0.887}\sqrt{0.887 - \frac{(k+1)^2}{8 \cdot 2\left((k+1)^2+3\right)^2}} - 1\right) > \delta \tag{3.1}$$

Let $t = k+1$. Computing the derivative of $f(t) = \frac{1}{0.887}\sqrt{0.887 - \frac{t^2}{8 \cdot 2(t^2+3)^2}}$ we see that it is positive, hence f is an increasing function. Evaluating f at the minimal possible $t = 3$ (i.e. $k = 2$), we get $f(2) \approx 1.0594$. Hence the left hand side of the inequality (3.1) is an increasing function. For the minimal $k = 2$ on the left hand side of (3.1) we get a number slightly bigger than 0.178 (the difference is on the fourth decimal place). Thus the inequality holds for each $k \geq 2$, if the round down of δ to the third decimal place is at most 0.178.

We have proved that $\varepsilon(L_S, r) > k+1+\delta$ for $\delta \in (0,\ 0.178]$. Therefore N is ample. □

Lemma 3.4. *There does not exist an effective divisor D such that*

$$ND - k - 1 \leq D^2 < \frac{ND}{2} < k+1.$$

Proof of the lemma. Assume that such a divisor exists. Then $D = \pi^* D_S - \sum_{i=1}^r m_i E_i$, where $m_i := \operatorname{mult}_{x_i} D_S$. Without loss of generality $D_S \not\equiv 0$. We consider two cases:

(1) $D^2 > 0$,
(2) $D^2 \leq 0$.

Ad. (1). By assumptions of the main theorem

$$r \leq 0.887 \cdot \frac{L_S^2}{(k+1)^2}$$

$$0.113 \cdot L_S^2 \leq L_S^2 - r \cdot (k+1)^2 = N^2$$

Since N is ample, by Hodge Index Theorem $N^2 D^2 \leq (ND)^2$. Obviously, $N^2 \leq N^2 D^2$. By assumption that a divisor D exists, $\frac{ND}{2} < k+1$. Moreover, $L_S^2 \geq 2((k+1)^2+3)^2$.

Altogether we get

$$0.113 \cdot 2((k+1)^2+3)^2 \leq 0.113 \cdot L_S^2 \leq N^2 \leq (ND)^2 \leq (2k+1)^2.$$

Therefore we have a series of inequalities

$$4k^2+4k+1 \geq 0.113 \cdot 2(k^4+4k^3+12k^2+16k+16) \geq$$

$$0.226 \cdot (4k^2+8k^2+12k^2+16k+16) \geq 0.226 \cdot (23k^2+18k+16) > 5k^2+4k+3,$$

which gives a contradiction in case $D^2 > 0$.

Ad. (2). $D^2 \leq 0$.

Since N is ample, $ND > 0$. Hence $ND \geq 1$. We also have that $ND - k - 1 \leq D^2$. Therefore

$$D^2 \geq ND - k - 1 \geq -k.$$

As $D = \pi^* D_S - \sum_{i=1}^r m_i E_i$, we have $D^2 = D_S^2 - \left(\sum_{i=1}^r m_i\right)^2$. Thus

$$-k \leq D_S^2 - \left(\sum_{i=1}^r m_i\right)^2.$$

Since $ND - k - 1 \leq D^2$ and $D^2 \leq 0$, we get that $ND \leq k+1$. We compute:

$$ND = \left(\pi^* L_S - (k+1)\sum_{i=1}^r E_i\right) \cdot \left(\pi^* D_S - \sum_{i=1}^r m_i E_i\right) = L_S D_S - (k+1)\sum_{i=1}^r m_i.$$

Therefore

$$L_S D_S = ND + (k+1)\sum_{i=1}^r m_i \leq (k+1)\left(1 + \sum_{i=1}^r m_i\right).$$

Since $\left(\sum_{i=1}^r m_i\right)^2 \leq D_S^2 + k$, we have

$$L_S D_S \leq (k+1)\left(1 + \sum_{i=1}^r m_i\right) \leq (k+1)\left(1 + D_S^2 + k\right).$$

Clearly, $D_S^2 \geq 0$.

If $D_S^2 = 0$, then $L_S D_S \leq (k+1)^2$. On the other hand, D_S is effective and $D_S \not\equiv 0$, hence if $D_S \equiv (\alpha, \beta)$, where $\alpha \geq 0$, $\beta \geq 0$ and α or β non-zero, then

$$L_S D_S = a\beta + b\alpha \geq \min\{a, b\} \geq d+2.$$

Therefore

$$(k+1)^2 + 3 \leq d + 2 \leq L_S D_S \leq (k+1)^2,$$

a contradiction.

If $D_S^2 > 0$, then by $L_S D_S \leq (k+1)(1 + D_S^2 + k)$ and Hodge Index Theorem we get

$$\begin{aligned} & L_S^2 D_S^2 \leq (L_S D_S)^2 \leq (k+1)^2 \left(1 + D_S^2 + k\right)^2 \\ & \| \\ & 2abD_S \geq 2(d+2)^2 \cdot D_S^2 \geq 2((k+1)^2+2)^2 \cdot D_S^2 \geq 2(k+1)^4 \cdot D_S^2. \end{aligned}$$

Hence

$$2(k+1)^2 \cdot D_S^2 \leq \left(k+1+D_S^2\right)^2.$$

We denote $z = D_S^2$, $t = k+1$. We have

$$2t^2 z \leq (t+z)^2,$$

$$0 \leq z^2 + \left(2t - 2t^2\right) z + t^2,$$

which is a quadratic equation in the variable z. Let $z_1(t) = -t + t^2 - \sqrt{t^4 - 2t^3}$, $z_2(t) = -t + t^2 + \sqrt{t^4 - 2t^3}$ be the roots of the equation. We will show that the open interval $(z_1(t), z_2(t))$ contains the closed interval $\left[1, \frac{1000}{887} t^2\right]$.

We compute the derivative: $z_1'(t) = -1 + 2t - \frac{3t^2 - 2t^3}{\sqrt{t^4 - 2t^3}}$. It is easy to verify that $z_1'(t) < 0$ for all admissible $t \geq 3$, hence $z_1(t)$ is a decreasing function. Evaluating z_1 at the minimal possible $t = 3$ $(k = 2)$, we get $z_1(3) \approx 0.804 < 1$.

Now we compute the derivative: $z_2'(t) - \frac{1000}{887} t^2 = -1 + 2t + \frac{3t^2 - 2t^3}{\sqrt{t^4 - 2t^3}} - \frac{1000}{887} t^2$. It is greater than 0 for all admissible $t \geq 3$, so $z_2(t) - \frac{1000}{887} t^2$ is an increasing function. Evaluating at the minimal possible $t = 3$ $(k = 2)$, we get the value of approximately $0.001 > 0$.

Thus we have a contradiction for $0 < D_S^2 \leq \frac{1000}{887}(k+1)^2$.

Let $D_S^2 > \frac{1000}{887}(k+1)^2$. By definition of the multi-point Seshadri constant

$$\varepsilon(L_S, r) \cdot \sum_{i=1}^{r} m_i \leq L_S D_S.$$

We have already proved that $\varepsilon(L_S, r) \geq k + 1 + \delta$ so

$$L_S D_S \geq \varepsilon(L_S, r) \cdot \sum_{i=1}^{r} m_i \geq (k+1+\delta) \sum_{i=1}^{r} m_i.$$

On the other hand, we have shown that $L_S D_S \leq (k+1)\left(1 + \sum_{i=1}^{r} m_i\right)$, therefore

$$(k+1)\left(1 + \sum_{i=1}^{r} m_i\right) \geq (k+1+\delta) \sum_{i=1}^{r} m_i.$$

Setting $t = k+1$, we have

$$t\left(1 + \sum_{i=1}^{r} m_i\right) \geq (t+\delta) \sum_{i=1}^{r} m_i.$$

$$\frac{1}{\delta} t \geq \sum_{i=1}^{r} m_i.$$

Thus

$$L_S D_S \leq t\left(1 + \sum_{i=1}^{r} m_i\right) \leq t\left(1 + \frac{1}{\delta} t\right).$$

Squaring both sides we get

$$(L_S D_S)^2 \leq t^2 \left(1 + \frac{1}{\delta} t\right)^2.$$

By Hodge Index Theorem and assumptions $(L_S D_S)^2 \geq L_S^2 D_S^2 > 2\left(t^2+3\right)^2 \frac{1000}{887} t^2$, hence

$$2\left(t^2+3\right)^2 \frac{1000}{887} t^2 - t^2 \left(1 + \frac{1}{\delta} t\right)^2 < 0,$$

$$2\left(t^2+3\right)^2 \frac{1000}{887} - \left(1 + \frac{1}{\delta} t\right)^2 < 0. \tag{3.2}$$

We set the maximal possible by previous computations $\delta = 0.178$ and compute the derivative of $g(t) = 2\left(t^2+3\right)^2 \frac{1000}{887} - \left(1 + \frac{1}{0.178} t\right)^2$. We obtain $g'(t) > 0$ for $t \geq 3$, hence g is an increasing function for $t \geq 3$. Since the value of g for the minimal possible $t = 3$ ($k = 2$) is positive, we get a contradiction. □

We have shown that by Theorem 2.6 the divisor $K_{\widetilde{S}} + N$ is k-very ample, but $K_{\widetilde{S}} + N = L$. □

We conclude with a remark.

Remark 3.5. *If we improved an estimation of multi-point Seshadri constant of a line bundle on a hyperelliptic surface, then we could easily show that the assertion of main theorem is satisfied with a bigger constant c, and therefore the line bundle L is k-very ample on the blow-up of a hyperelliptic surface in more very general points.*

However, if we want to apply Theorem 2.6 to $L - K_{\widetilde{S}}$ then a constant c, rounded down to the third decimal place, cannot exceed the number 0.954*, as otherwise the inequality $N^2 \geq 4k+5$ would not hold.*

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